

# On the $\eta$ - $\eta'$ complex in the SD–BS approach

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## Abstract

The bound-state Schwinger-Dyson and Bethe-Salpeter (SD–BS) approach is chirally well-behaved and provides a reliable treatment of the  $\eta$ - $\eta'$  complex although a ladder approximation is employed. Allowing for the effects of the SU(3) flavor symmetry breaking in the quark–antiquark annihilation, leads to the improved  $\eta$ - $\eta'$  mass matrix.

## I. $\eta$ - $\eta'$ PHENOMENOLOGY AND GOLDSTONE STRUCTURE

The physical isoscalar pseudoscalars  $\eta$  and  $\eta'$  are usually given as

$$|\eta\rangle = \cos\theta |\eta_8\rangle - \sin\theta |\eta_0\rangle, \quad |\eta'\rangle = \sin\theta |\eta_8\rangle + \cos\theta |\eta_0\rangle, \quad (1)$$

*i.e.*, as the orthogonal mixture of the respective octet and singlet isospin zero states,  $\eta_8$  and  $\eta_0$ . In the flavor SU(3) quark model, they are defined through quark–antiquark ( $q\bar{q}$ ) basis states  $|f\bar{f}\rangle$  ( $f = u, d, s$ ) as

$$|\eta_8\rangle = \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle), \quad (2a)$$

$$|\eta_0\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle). \quad (2b)$$

The exact SU(3) flavor symmetry ( $m_u = m_d = m_s$ ) is nevertheless badly broken. It is an excellent approximation to assume the exact isospin symmetry ( $m_u = m_d$ ), and a good approximation to take even the chiral symmetry limit for  $u$  and  $d$ -quark (where *current* quark masses  $m_u = m_d = 0$ ), but for a realistic description, the strange quark mass  $m_s$  must be significantly heavier than  $m_u$  and  $m_d$ . The same holds for the *constituent* quark masses, denoted by  $\hat{m}$  for *both*  $u$  and  $d$  quarks since we rely on the isosymmetric limit, and by  $\hat{m}_s$  for the  $s$ -quark. They are nonvanishing in the chiral limit (CL). In the strange sector, CL is useful only qualitatively, as a theoretical limit. (CL would reduce  $\hat{m}_s$  to  $\hat{m}$ , on which CL has almost negligible influence.)

Thus, with  $|u\bar{u}\rangle$  and  $|d\bar{d}\rangle$  being practically chiral states as opposed to a significantly heavier  $|s\bar{s}\rangle$ , Eqs. (2) do not define the octet and singlet states of the exact SU(3) flavor

symmetry, but the *effective* octet and singlet states. Hence, as in Ref. [1] for example, only in the sense that the same  $q\bar{q}$  states  $|f\bar{f}\rangle$  ( $f = u, d, s$ ) appear in both Eq. (2a) and Eq. (2b) do these equations implicitly assume nonet symmetry (as pointed out by Gilman and Kauffman [2], following Chanowitz, their Ref. [8]). However, in order to avoid the  $U_A(1)$  problem, this symmetry must ultimately be broken at least at the level of the masses. In particular, it must be broken in such a way that  $\eta \rightarrow \eta_8$  becomes massless but  $\eta' \rightarrow \eta_0$  remains massive (as in Ref. [1]) when CL is taken for all three flavors,  $m_u, m_d, m_s \rightarrow 0$ . Nevertheless, the CL-vanishing octet eta mass  $m_{\eta_8}$  is rather heavy for the realistically broken  $SU(3)$  flavor symmetry; for the empirical pion and kaon masses  $m_\pi$  and  $m_K$ , the Gell-Mann-Okubo mass formula  $m_\pi^2 + 3m_{\eta_8}^2 = 4m_K^2$  yields  $m_{\eta_8} \approx 567$  MeV. In that case, and for the empirical masses of  $\eta(547)$  and  $\eta'(958)$ , the singlet  $\eta_0$  mass  $m_{\eta_0}$  (nonvanishing even in CL) can be found from the mass-matrix trace

$$m_{\eta_8}^2 + m_{\eta_0}^2 = m_\eta^2 + m_{\eta'}^2 \approx 1.22 \text{ GeV}^2, \quad \text{giving } m_{\eta_0} \approx 947 \text{ MeV}. \quad (3)$$

Alternatively, one can work in a nonstrange ( $NS$ )–strange ( $S$ ) basis:

$$|\eta_{NS}\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}}|\eta_8\rangle + \sqrt{\frac{2}{3}}|\eta_0\rangle, \quad (4a)$$

$$|\eta_S\rangle = |s\bar{s}\rangle = -\sqrt{\frac{2}{3}}|\eta_8\rangle + \frac{1}{\sqrt{3}}|\eta_0\rangle. \quad (4b)$$

In analogy with Eq. (3), in this basis one finds

$$m_{\eta_{NS}}^2 + m_{\eta_S}^2 = m_\eta^2 + m_{\eta'}^2 \approx 1.22 \text{ GeV}^2, \quad (5)$$

whereas the  $NS$ – $S$  mixing relations, diagonalizing the mass matrix, are

$$|\eta\rangle = \cos\phi_P|\eta_{NS}\rangle - \sin\phi_P|\eta_S\rangle, \quad |\eta'\rangle = \sin\phi_P|\eta_{NS}\rangle + \cos\phi_P|\eta_S\rangle. \quad (6)$$

The singlet-octet mixing angle  $\theta$ , defined by Eqs. (1), is related to the  $NS$ – $S$  mixing angle  $\phi$  above as [3]  $\theta = \phi - \arctan\sqrt{2} = \phi - 54.74^\circ$ .

Although mathematically equivalent to the  $\eta_8$ – $\eta_0$  basis, the  $NS$ – $S$  mixing basis is more suitable for most quark model considerations, being more natural in practice when the symmetry between the  $NS$  and  $S$  sectors is broken as described in the preceding passage. There is also another important reason to keep in mind the  $|\eta_{NS}\rangle$ – $|\eta_S\rangle$  state mixing angle  $\phi$ . This is because it offers the quickest way to show the consistency of our procedures and the corresponding results obtained using just one ( $\theta$  or  $\phi$ ) *state* mixing angle, with the two-mixing-angle scheme considered in Refs. [4–10], which is defined with respect to the mixing of the decay constants. For clarification of the relationship with, and our results in the two-mixing-angle scheme, we refer to Ref. [11], particularly to its Appendix. Here, we simply note that our considerations will ultimately lead us to  $\phi \approx 42^\circ$ , practically the same as the result of Refs. [6–8,10] and in agreement with data (*e.g.*, see Table 2 of Feldmann’s review [10]).

# FIGURES

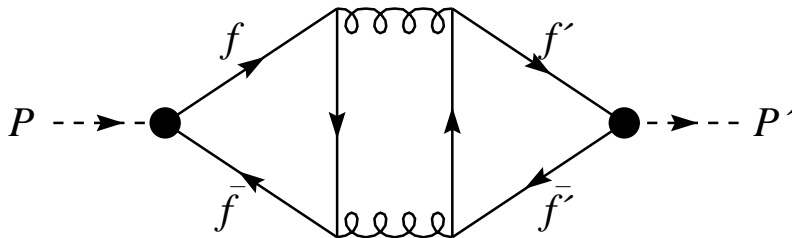


FIG. 1. Nonperturbative QCD quark annihilation illustrated by the two-gluon exchange diagram. It shows the transition of the  $f\bar{f}$  pseudoscalar  $P$  into the pseudoscalar  $P'$  having the flavor content  $f'\bar{f}'$ . The dashed lines and full circles depict the  $q\bar{q}$  bound-state pseudoscalars and vertices, respectively.

As for a theoretical determination of the  $\eta$ - $\eta'$  mixing angle  $\phi$  or  $\theta$ , we follow the path of Refs. [3]. The contribution of the gluon axial anomaly to the singlet  $\eta_0$  mass is essentially just parameterized and not really calculated, but some useful information can be obtained from the isoscalar  $q\bar{q}$  annihilation graphs of which the “diamond” one in Fig. 1 is just the simplest example. That is, we can take Fig. 1 in the nonperturbative sense, where the two-gluon intermediate “states” represent any even number of gluons when forming a  $C^+$  pseudoscalar  $\bar{q}q$  meson [12], and where quarks, gluons and vertices can be dressed nonperturbatively, and possibly include gluon configurations such as instantons. Factorization of the quark propagators in Fig. 1 characterized by the ratio  $X \approx \hat{m}/\hat{m}_s$  leads to the  $\eta$ - $\eta'$  mass matrix in the  $NS$ - $S$  basis [3]

$$\begin{pmatrix} m_\pi^2 + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & 2m_K^2 - m_\pi^2 + \beta X^2 \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} m_\eta^2 & 0 \\ 0 & m_{\eta'}^2 \end{pmatrix}, \quad (7)$$

where  $\beta$  denotes the total annihilation strength of the pseudoscalar  $q\bar{q}$  for the *light* flavors  $f = u, d$ , whereas it is assumed attenuated by a factor  $X$  when a  $s\bar{s}$  pseudoscalar appears. (The mass matrix in the  $\eta_8$ - $\eta_0$  basis reveals that in the  $X \rightarrow 1$  limit, the CL-nonvanishing singlet  $\eta_0$  mass is given by  $3\beta$ .) The two parameters on the left-hand-side (LHS) of (7),  $\beta$  and  $X$ , are determined by the two diagonalized  $\eta$  and  $\eta'$  masses on the RHS of (7). The trace and determinant of the matrices in (7) then fix  $\beta$  and  $X$  to be [3]

$$\beta = \frac{(m_{\eta'}^2 - m_\pi^2)(m_\eta^2 - m_\pi^2)}{4(m_K^2 - m_\pi^2)} \approx 0.28 \text{ GeV}^2, \quad X \approx 0.78, \quad (8)$$

with the latter value suggesting a  $S/NS$  constituent quark mass ratio  $X^{-1} \sim \hat{m}_s/\hat{m} \sim 1.3$ , near the values in Refs. [12–16],  $\hat{m}_s/\hat{m} \approx 1.45$ .

This fitted nonperturbative scale of  $\beta$  in (8) depends only on the gross features of QCD. If instead one treats the QCD graph of Fig. 1 in the perturbative sense of literally two gluons exchanged, then one obtains [17] only  $\beta_{2g} \sim 0.09 \text{ GeV}^2$ , which is about 1/3 of the needed scale of  $\beta$  found in (8). (This indicates that just the perturbative “diamond” graph can

hardly represent even the roughest approximation to the effect of the gluon axial anomaly operator  $\epsilon^{\alpha\beta\mu\nu} G_{\alpha\beta}^a G_{\mu\nu}^a$ .) The above fitted quark annihilation (nonperturbative) scale  $\beta$  in (8) can be converted to the  $NS$ - $S$   $\eta$ - $\eta'$  mixing angle  $\phi$  in (6) from the alternative mixing relation  $\tan 2\phi = 2\sqrt{2}\beta X(m_{\eta_S}^2 - m_{\eta_{NS}}^2)^{-1} \approx 9.02$  to [3]

$$\phi = \arctan \left[ \frac{(m_{\eta'}^2 - 2m_K^2 + m_\pi^2)(m_\eta^2 - m_\pi^2)}{(2m_K^2 - m_\pi^2 - m_\eta^2)(m_{\eta'}^2 - m_\pi^2)} \right]^{1/2} \approx 41.84^\circ. \quad (9)$$

This kinematical QCD mixing angle (9) or  $\theta = \phi - 54.74^\circ \approx -12.9^\circ$  has dynamical analogs [1,11], namely the coupled SD-BS approach discussed below, in Sec. II. Since this predicted  $\eta$ - $\eta'$  mixing angle in (9) is compatible with the values repeatedly extracted in various empirical ways [13,14], and more recently from the FKS scheme and theory [6–10], we confidently use the value (9) in the mixing angle relations (6) to infer the nonstrange and strange  $\eta$  masses,

$$m_{\eta_{NS}}^2 = \cos^2 \phi \, m_\eta^2 + \sin^2 \phi \, m_{\eta'}^2 \approx (757.9 \text{ MeV})^2 \quad (10a)$$

$$m_{\eta_S}^2 = \sin^2 \phi \, m_\eta^2 + \cos^2 \phi \, m_{\eta'}^2 \approx (801.5 \text{ MeV})^2. \quad (10b)$$

Thus it is clear that the true physical masses  $\eta(547)$  and  $\eta'(958)$  are respectively much closer to the Nambu-Goldstone (NG) octet  $\eta_8(567)$  and the non-NG singlet  $\eta_0(947)$  configurations than to the nonstrange  $\eta_{NS}(758)$  and strange  $\eta_S(801)$  configurations inferred in Eqs. (10). However, the mean  $\eta$ - $\eta'$  mass  $(548 + 958)/2 \approx 753$  MeV is quite near the nonstrange  $\eta_{NS}(758)$ . But since  $\eta_8(567)$  appears far from the NG massless limit we must ask: how close is  $\eta_0(947)$  to the chiral-limiting nonvanishing singlet  $\eta$  mass?

To answer this latter question, return to Fig. 1 and the quark annihilation strength  $\beta \approx 0.28 \text{ GeV}^2$  in Eq. (8). These  $\bar{q}q$  states presumably hadronize into the  $U_A(1)$  singlet state (2b), for effective squared mass in the  $SU(3)$  limit with  $\beta$  remaining unchanged [17]:

$$m_{\eta_0}^2 = 3\beta \approx (917 \text{ MeV})^2. \quad (11)$$

This latter CL  $\eta_0$  mass in (11) is only 3% shy of the exact chiral-broken  $\eta_0(947)$  mass found in Eq. (3). (Such a 3% CL reduction also holds for the pion decay constant  $f_\pi \approx 93 \text{ MeV} \rightarrow 90 \text{ MeV}$  [18] and for  $f_+(0) = 1 \rightarrow 0.97$  [19], the  $K-\pi$   $K_{l3}$  form factor.)

Our  $\eta$ - $\eta'$  mixing analysis on the basis of phenomenological mass inputs thus tells us that the physical  $\eta(547)$  is 97% of the *chiral-broken* NG boson  $\eta_8(567)$ . Also the mixing-induced CL singlet mass of 917 MeV in (11) is 97% of the chiral-broken singlet  $\eta_0(947)$  in (3), which in turn is 99% of the physical  $\eta'$  mass  $\eta'(958)$ . This can be viewed as the phenomenological resolution of the  $U_A(1)$  problem of the masses and (quasi-)Goldstone boson structure of the observed  $\eta(547)$  and  $\eta'(958)$  mesons. Or rather, from a more microscopic standpoint, the above represents phenomenological constraints that microscopic, more or less QCD-based studies of the  $\eta$ - $\eta'$  complex must respect.

## II. BOUND-STATE SD-BS APPROACH TO $\eta$ - $\eta'$

The coupled Schwinger-Dyson (SD) and Bethe-Salpeter (BS) approach [20] can be formulated so that it has strong and clear connections with QCD, the fundamental theory of strong

interactions. In this approach, by solving the SD equation for dressed quark propagators of various flavors, one explicitly constructs constituent quarks. They in turn build  $q\bar{q}$  meson bound states which are solutions of the BS equation employing the dressed quark propagator obtained as the solution of the SD equation. If the SD and BS equations are so coupled in a consistent approximation, the light pseudoscalar mesons are simultaneously the  $q\bar{q}$  bound states and the (quasi) Goldstone bosons of dynamical chiral symmetry breaking (D $\chi$ SB). The resulting relativistically covariant bound-state model (such as the variant of Ref. [21]) is consistent with current algebra because it incorporates the correct chiral symmetry behavior thanks to D $\chi$ SB obtained in an, essentially, Nambu–Jona-Lasinio fashion, but the SD–BS model interaction is less schematic. In Refs. [1,21–25] for example, it is combined nonperturbative and perturbative gluon exchange; the effective propagator function is the sum of the known perturbative QCD contribution and the modeled nonperturbative component. For details, we refer to Refs. [1,21–24], while here we just note that the momentum-dependent dynamically generated quark mass functions  $\mathcal{M}_f(q^2)$  (*i.e.*, the quark propagator SD solutions for quark flavors  $f$ ) illustrate well how the coupled SD-BS approach provides a modern constituent model which is consistent with perturbative and nonperturbative QCD. For example, the perturbative QCD part of the gluon propagator leads to the deep Euclidean behaviors of quark propagators (for all flavors) consistent with the asymptotic freedom of QCD [23]. However, what is important in the present paper, is the behavior of the mass functions  $\mathcal{M}_f(q^2)$  for *low* momenta [ $q^2 = 0$  to  $-q^2 \approx (400 \text{ MeV})^2$ ], where  $\mathcal{M}_f(q^2)$  (due to D $\chi$ SB) has values consistent with typical values of the constituent mass parameter in constituent quark models. For the (isosymmetric)  $u$ - and  $d$ -quarks, our concrete model choice [21] gives us  $\mathcal{M}_{u,d}(0) = 356 \text{ MeV}$  in the chiral limit (*i.e.*, with vanishing  $\widetilde{m}_{u,d}$ , the explicit chiral symmetry breaking bare mass term in the quark propagator SD equation, resulting in vanishing pion mass eigenvalue,  $m_\pi = 0$ , in the BS equation), and  $\mathcal{M}_{u,d}(0) = 375 \text{ MeV}$  [just 5% above  $\mathcal{M}_{u,d}(0)$  in the chiral limit] with the bare mass  $\widetilde{m}_{u,d} = 3.1 \text{ MeV}$ , leading to a realistically light pion,  $m_\pi = 140.4 \text{ MeV}$ . Similarly, for the  $s$  quark,  $\mathcal{M}_s(0) = 610 \text{ MeV}$ . The simple-minded constituent masses in both  $NS$  and  $S$  sectors,  $\hat{m}$  and  $\hat{m}_s$  employed in Sec. I, have thus close analogues in the coupled SD–BS approach which explicitly incorporates some crucial features of QCD, notably D $\chi$ SB. Thanks to D $\chi$ SB, this dynamical, bound-state approach successfully incorporates the partially Goldstone boson structure of the mixed  $\eta(547)$  and  $\eta'(958)$  mesons [1].

Before addressing its mass matrix, let us briefly recall what the SD–BS approach revealed [1,11] about the mixing angle inferred from  $\eta, \eta' \rightarrow \gamma\gamma$  decays. The SD–BS approach incorporates the correct chiral symmetry behavior thanks to D $\chi$ SB and is consistent with current algebra. Therefore, and this gives particular weight to the constraints placed on the mixing angle  $\theta$  by the SD-BS results on  $\gamma\gamma$  decays of pseudoscalars, this approach reproduces (when care is taken to preserve the vector Ward-Takahashi identity of QED) analytically and exactly the CL pseudoscalar  $\rightarrow \gamma\gamma$  decay amplitudes (*e.g.*,  $\pi^0 \rightarrow \gamma\gamma$ ), which are fixed by the Abelian axial anomaly. (Note that they are otherwise notoriously difficult to reproduce in bound-state approaches, as discussed in Ref. [23].)

General and robust considerations in this chirally well-behaved approach showed [1] that, unlike the pion case,  $\eta_8, \eta_0 \rightarrow \gamma\gamma$  (and therefore also their mixtures  $\eta, \eta' \rightarrow \gamma\gamma$ ) decay amplitudes cannot be given through their respective axial-current decay constants  $f_{\eta_8}, f_{\eta_0}$ , and also gave strong bounds on these amplitudes with respect to the pion decay constant

$f_\pi$  (*i.e.*, w. r. to the  $\pi^0 \rightarrow \gamma\gamma$  amplitude). All this says that in models relying on quark degrees of freedom, reasonably accurate reproduction of the empirical  $\eta, \eta' \rightarrow \gamma\gamma$  widths is possible only for  $\theta$ -values less negative than  $-15^\circ$ . For the concrete [22,21] model adopted in Ref. [1], our calculated  $\eta, \eta' \rightarrow \gamma\gamma$  widths fit the data best for  $\theta = -12.0^\circ$ .

For the very predictive SD-BS approach to be consistent, the above mixing angle extracted from  $\eta, \eta' \rightarrow \gamma\gamma$  widths, should be close to the angle  $\theta$  predicted by diagonalizing the  $\eta$ - $\eta'$  mass matrix. In this section, it is given in the quark  $f\bar{f}$  basis:

$$\hat{M}^2 = \text{diag}(M_{u\bar{u}}^2, M_{d\bar{d}}^2, M_{s\bar{s}}^2) + \beta \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (12)$$

As in Sec. I,  $3\beta$  (called  $\lambda_\eta$  in Ref. [1]) is the contribution of the gluon axial anomaly to  $m_{\eta_0}^2$ , the squared mass of  $\eta_0$ . We denote by  $M_{f\bar{f}'}$  the masses obtained as eigenvalues of the BS equations for  $q\bar{q}$  pseudoscalars with the flavor content  $f\bar{f}'$  ( $f, f' = u, d, s$ ). However, since Ref. [1] had to employ a rainbow-ladder approximation (albeit the improved one of Ref. [21]), it could not calculate the gluon axial anomaly contribution  $3\beta$ . It could only avoid the  $U_A(1)$ -problem in the  $\eta$ - $\eta'$  complex by *parameterizing*  $3\beta$ , namely that part of the  $\eta_0$  mass squared which remains nonvanishing in the CL. Because of the rainbow-ladder approximation (which does not contain even the simplest annihilation graph – Fig. 1), the  $q\bar{q}$  pseudoscalar masses  $M_{f\bar{f}'}$  *do not* contain any contribution from  $3\beta$ , unlike the nonstrange and strange  $\eta$  masses  $m_{\eta_{NS}}$  [in Eq. (10a)] and  $m_{\eta_S}$  [in Eq. (10b)], which do, and which must not be confused with  $M_{u\bar{u}} = M_{d\bar{d}}$  and  $M_{s\bar{s}}$ . Since the flavor singlet gluon anomaly contribution  $3\beta$  does not influence the masses  $m_\pi$  and  $m_K$  of the non-singlet pion and kaon, the realistic rainbow-ladder modeling aims directly at reproducing the empirical values of these masses:  $M_{u\bar{u}} = M_{d\bar{d}} = m_\pi$  and  $M_{s\bar{d}} = m_K$ . In contrast, the masses of the physical etas,  $m_\eta$  and  $m_{\eta'}$ , must be obtained by diagonalizing the  $\eta_8$ - $\eta_0$  sub-matrix containing both  $M_{f\bar{f}}$  and the gluon anomaly contribution to  $m_{\eta_0}^2$ .

Since the gluon anomaly contribution  $3\beta$  vanishes in the large  $N_c$  limit as  $1/N_c$ , while all  $M_{f\bar{f}'}$  vanish in CL, our  $q\bar{q}$  bound-state pseudoscalar mesons behave in the  $N_c \rightarrow \infty$  and chiral limits in agreement with QCD and  $\chi$ PT (*e.g.*, see [26]): as the strict CL is approached for all three flavors, the SU(3) octet pseudoscalars *including*  $\eta$  become massless Goldstone bosons, whereas the chiral-limit-nonvanishing  $\eta'$ -mass  $3\beta$  is of order  $1/N_c$  since it is purely due to the gluon anomaly. If one lets  $3\beta \rightarrow 0$  (as the gluon anomaly contribution behaves for  $N_c \rightarrow \infty$ ), then for any quark masses and resulting  $M_{f\bar{f}}$  masses, the “ideal” mixing ( $\theta = -54.74^\circ$ ) takes place so that  $\eta$  consists of  $u, d$  quarks only and becomes degenerate with  $\pi$ , whereas  $\eta'$  is the pure  $s\bar{s}$  pseudoscalar bound state with the mass  $M_{s\bar{s}}$ .

In Ref. [1], numerical calculations of the mass matrix were performed for the realistic chiral and SU(3) symmetry breaking, with the finite quark masses (and thus also the finite BS  $q\bar{q}$  bound-state pseudoscalar masses  $M_{f\bar{f}}$ ) fixed by the fit [21] to static properties of many mesons but excluding the  $\eta$ - $\eta'$  complex. The mixing angle which diagonalizes the  $\eta_8$ - $\eta_0$  mass matrix thus depended in Ref. [1] only on the value of the additionally introduced “gluon anomaly parameter”  $3\beta$ . Its preferred value turned out to be  $3\beta = 1.165 \text{ GeV}^2 = (1079 \text{ MeV})^2$ , leading to the mixing angle  $\theta = -12.7^\circ$  [compatible with  $\phi = 41.84^\circ$  in Eq. (9)] and acceptable  $\eta \rightarrow \gamma\gamma$  and  $\eta' \rightarrow \gamma\gamma$  decay amplitudes. Also, the  $\eta$  mass was then fitted to its experimental value, but such a high value of  $3\beta$  inevitably resulted in a too high  $\eta'$

mass, above 1 GeV. (Conversely, lowering  $3\beta$  aimed to reduce  $m_{\eta'}$ , would push  $\theta$  close to  $-20^\circ$ , making predictions for  $\eta, \eta' \rightarrow \gamma\gamma$  intolerably bad.) However, unlike Eq. (7) in the present paper, it should be noted that Ref. [1] did not introduce into the mass matrix the “strangeness attenuation parameter”  $X$  which should suppress the nonperturbative quark  $f\bar{f} \rightarrow f'\bar{f}'$  annihilation amplitude (illustrated by the “diamond” graph in Fig. 1) when  $f$  or  $f'$  are strange. Ref. [11] concluded that it was precisely the lack of the strangeness attenuation factor  $X$  that prevented Ref. [1] from satisfactorily reproducing the  $\eta'$  mass when it successfully did so with the  $\eta$  mass and  $\gamma\gamma$  widths.

One can expect that the influence of this suppression should be substantial, since  $X \approx \hat{m}/\hat{m}_s$  should be a reasonable estimate of it, and this nonstrange-to-strange *constituent* mass ratio in the considered variant of the SD-BS approach [1] is not far from  $X$  in Eq. (8) and from the mass ratios in Refs. [12,15,16], and is even closer to the mass ratios in the Refs. [14]. Namely, two of us found [1] it to be around  $\mathcal{M}_u(0)/\mathcal{M}_s(0) = 0.615$  if the constituent mass was defined at the vanishing argument  $q^2$  of the momentum-dependent SD mass function  $\mathcal{M}_f(q^2)$ .

We therefore introduce the suppression parameter  $X$  the same way as in the  $NS$ – $S$  mass matrix (7), whereby the mass matrix in the  $f\bar{f}$  basis becomes

$$\hat{M}^2 = \text{diag}(M_{u\bar{u}}^2, M_{d\bar{d}}^2, M_{s\bar{s}}^2) + \beta \begin{pmatrix} 1 & 1 & X \\ 1 & 1 & X \\ X & X & X^2 \end{pmatrix}. \quad (13)$$

The very accurate isospin symmetry makes the mixing of the isovector  $\pi^0$  and the isoscalar etas negligible for all our practical purposes. Going to a meson basis of  $\pi^0$  and etas enables us therefore to separate the  $\pi^0$  and restrict  $\hat{M}^2$  to the  $2 \times 2$  subspace of the etas. In the  $NS$ – $S$  basis,

$$\begin{pmatrix} m_{\eta_{NS}}^2 & m_{\eta_S \eta_{NS}}^2 \\ m_{\eta_{NS} \eta_S}^2 & m_{\eta_S}^2 \end{pmatrix} = \begin{pmatrix} M_{u\bar{u}}^2 + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M_{s\bar{s}}^2 + \beta X^2 \end{pmatrix}. \quad (14)$$

To a very good approximation, Eq. (14) recovers Eq. (7). This is because not only  $m_\pi = M_{u\bar{u}} = M_{d\bar{d}}$ , but also because  $M_{s\bar{s}}^2$  differs from  $2m_K^2 - m_\pi^2$  only by a couple of percent, thanks to the good chiral behavior of the masses  $M_{f\bar{f}'}$  calculated in SD-BS approach. (These  $M_{f\bar{f}'}$  and the CL model values of  $f_\pi$  and quark condensate, satisfy Gell-Mann-Oakes-Renner relation to first order in the explicit chiral symmetry breaking [22].) The SD-BS–predicted octet (quasi-)Goldstone masses  $M_{f\bar{f}'}$  are known to be empirically successful in our concrete model choice [21], but the question is whether the SD-BS approach can also give some information on the  $X$ -parameter. If we treat *both*  $3\beta$  and  $X$  as free parameters, we can of course fit both the  $\eta$  mass and the  $\eta'$  mass to their experimental values. For the model parameters as in Ref. [21] (for these parameters our independent calculation gives  $m_\pi = M_{u\bar{u}} = 140.4$  MeV and  $M_{s\bar{s}} = 721.4$  MeV), this happens at  $3\beta = 0.753 \text{ GeV}^2 = (868 \text{ MeV})^2$  and  $X = 0.835$ . However, the mixing angle then comes out as  $\theta = -17.9^\circ$ , which is too negative to allow consistency of the empirically found two-photon decay amplitudes of  $\eta$  and  $\eta'$ , with predictions of our SD-BS approach for the two-photon decay amplitudes of  $\eta_8$  and  $\eta_0$  [1].

Therefore, and also to avoid introducing another free parameter in addition to  $3\beta$ , we take the path where the dynamical information from our SD-BS approach is used to estimate  $X$ . Namely, our  $\gamma\gamma$  decay amplitudes  $T_{f\bar{f}}$  can be taken as a serious guide for estimating the  $X$ -parameter instead of allowing it to be free. We did point out in Sec. I that the attempted treatment [17] of the gluon anomaly contribution through just the “diamond diagram” contribution to  $3\beta$ , indicated that just this partial contribution is quite insufficient. This limits us to keeping  $3\beta$  as a free parameter, but we can still suppose that this diagram can help us get the prediction of the strange-nonstrange *ratio* of the complete pertinent amplitudes  $f\bar{f} \rightarrow f'\bar{f}'$  as follows. Our SD-BS modeling in Ref. [1] employs an infrared-enhanced gluon propagator [21,23] weighting the integrand strongly for low gluon momenta squared. Therefore, in analogy with Eq. (4.12) of Kogut and Susskind [27] (see also Refs. [28,29]), we can approximate the Fig. 1 amplitudes  $f\bar{f} \rightarrow 2\text{gluons} \rightarrow f'\bar{f}'$ , *i.e.*, the contribution of the quark-gluon diamond graph to the element  $ff'$  of the  $3 \times 3$  mass matrix, by the factorized form

$$\tilde{T}_{f\bar{f}}(0,0) \mathcal{C} \tilde{T}_{f'\bar{f}'}(0,0). \quad (15)$$

In Eq. (15), the quantity  $\mathcal{C}$  is given by the integral over two gluon propagators remaining after factoring out  $\tilde{T}_{f\bar{f}}(0,0)$  and  $\tilde{T}_{f'\bar{f}'}(0,0)$ , the respective amplitudes for the transition of the  $q\bar{q}$  pseudoscalar bound state for the quark flavor  $f$  and  $f'$  into two vector bosons, in this case into two gluons. The contribution of Fig. 1 is thereby expressed with the help of the (reduced) amplitudes  $\tilde{T}_{f\bar{f}}(0,0)$  calculated in Ref. [1] for the transition of  $q\bar{q}$  pseudoscalars to two real photons ( $k^2 = k'^2 = 0$ ), while in general  $\tilde{T}_{f\bar{f}}(k^2, k'^2) \equiv T_{f\bar{f}}(k^2, k'^2)/Q_f^2$  are the “reduced” two-photon amplitudes obtained by removing the squared charge factors  $Q_f^2$  from  $T_{f\bar{f}}$ , the  $\gamma\gamma$  amplitude of the pseudoscalar  $q\bar{q}$  bound state of the hidden flavor  $f\bar{f}$ . Although  $\mathcal{C}$  is in principle computable, all this unfortunately does not amount to determining  $\beta, \beta X$  and  $\beta X^2$  in Eq. (13) since the higher (four-gluon, six-gluon, ... , etc.) contributions are clearly lacking. We therefore must keep the total (light-)quark annihilation strength  $\beta$  as a free parameter. However, if we assume that the suppression of the diagrams with the strange quark in a loop is similar for all of them, Eq. (15) and the “diamond” diagram in Fig. 1 help us to at least estimate the parameter  $X$  as  $X \approx \tilde{T}_{s\bar{s}}(0,0)/\tilde{T}_{u\bar{u}}(0,0)$ . This is a natural way to build in the effects of the SU(3) flavor symmetry breaking in the  $q\bar{q}$  annihilation graphs. (Recall that  $\tilde{T}_{s\bar{s}}(0,0)/\tilde{T}_{u\bar{u}}(0,0) \approx \hat{m}/\hat{m}_s$  to a good approximation [11].)

We get  $X = 0.663$  from the two-photon amplitudes we obtained in the chosen SD-BS model [21]. This value of  $X$  agrees well with the other way of estimating  $X$ , namely the nonstrange-to-strange constituent mass ratio of Refs. [12,15,16]. With  $X = 0.663$ , requiring that the  $2 \times 2$  matrix trace,  $m_\eta^2 + m_{\eta'}^2$ , be fitted to its empirical value, fixes the chiral-limiting nonvanishing singlet mass squared to  $3\beta = 0.832 \text{ GeV}^2 = (912 \text{ MeV})^2$ , just 0.5% below Eq. (11), while  $m_{\eta_{NS}} = 757.87 \text{ MeV}$  and  $m_{\eta_S} = 801.45 \text{ MeV}$ , practically the same as Eqs. (10). The resulting mixing angle and  $\eta, \eta'$  masses are

$$\phi = 41.3^\circ \text{ or } \theta = -13.4^\circ; \quad m_\eta = 588 \text{ MeV}, \quad m_{\eta'} = 933 \text{ MeV}. \quad (16)$$

These results are for the original parameters of Ref. [1]. Reference [11] also varied the parameters to check the sensitivity on SD-BS modeling, but the results changed little.

The above results of the SD-BS approach [1] are very satisfactory since they agree very well with what was found in Sec. I by different methods. They also agree with the UKQCD lattice results [30] on  $\eta$ - $\eta'$  mixing. Their calculated mixing parameter  $x_{ss}$  corresponds to our  $\beta X^2$ , and their mixing parameters  $x_{nn}$  and  $x_{ns}$  ( $n = u, d$ ), corresponding respectively to our  $\beta$  and  $\beta X$ , are aimed to obey  $x_{nn} \approx 2x_{ss}$  and  $x_{ns}^2 \approx x_{nn}x_{ss}$ . UKQCD prefers [30]  $x_{ss} = 0.13 \text{ GeV}^2$ ,  $x_{nn} = 0.292 \text{ GeV}^2$  and  $x_{ns} = 0.218 \text{ GeV}^2$ . This, together with their preferred *input* values  $M_{n\bar{n}} = 0.137 \text{ GeV}$  and  $M_{s\bar{s}} = 0.695 \text{ GeV}$ , give the  $NS$ - $S$  mass matrix (14) with elements reasonably close to ours, resulting in a rather close mixing angle,  $\theta = -10.2^\circ$ .

### III. CONCLUSION

We have shown that the treatment of the  $\eta$ - $\eta'$  complex in the SD-BS approach [1] is sensible in spite of employing the ladder approximation. This is confirmed especially by Ref. [11] which showed its connection and robust agreement with the phenomenological studies of the  $\eta$ - $\eta'$  complex. It is therefore desirable to extend the SD-BS studies of the  $\eta$ - $\eta'$  mass matrix to finite temperatures. Usually, one has neglected all temperature dependences in the mass matrix, except the one of the gluon anomaly contribution  $3\beta$  which is assumed very strong, which is appropriate if the  $U_A(1)$  symmetry breaking is due to instantons [31–34]. However, rather strong topological arguments of Kogut *et al.* [35] that the  $U_A(1)$  symmetry is not restored at critical (but only at a higher, possibly infinite)  $T$ , motivates also the scenario where  $3\beta(T) \approx \text{const}$ , while other entries in the mass matrix carry the temperature dependence. The inclusion of their  $T$ -dependence is needed also because the scenario with the instanton-induced, strongly  $T$ -dependent  $\beta$  should be carefully re-examined, since it has lead to contradicting conclusions: the depletion of  $\eta'$  production in Ref. [32], but  $\eta'$ -enhancement in Ref. [33].

The temperature dependence of  $m_\pi = M_{u\bar{u}} = M_{d\bar{d}}$ ,  $\mathcal{M}_{u,d}(q^2)$ ,  $f_\pi$  and  $\langle u\bar{u} \rangle (= \langle d\bar{d} \rangle)$ , was already studied in various SD-BS models [36,37], so that the extension [38] to the  $T$ -dependence of the remaining needed ingredients,  $M_{s\bar{s}}$ ,  $\mathcal{M}_s(q^2)$ ,  $f_{s\bar{s}}$  and  $\langle s\bar{s} \rangle$ , should be straightforward.

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